M381
Course Guide

CONTENTS

1 Course Guide 1
2 Mathematical Logic Guide 4
3 Errata and Addenda: Mathematical Logic 8

1 COURSE GUIDE

This course consists of two halves: Number Theory (abbreviated as NT), and Mathematical Logic (abbreviated as ML). The teaching material in one half is completely independent of the material in the other, so that they form separate mini-courses. This does not mean that there are no links between the subjects covered — in fact there are, and you will see how ML deals with problems in the foundations of Mathematics arising from Number Theory. One potential source of confusion is what is meant by the set of *natural numbers*. Standard practice amongst modern logicians is that this is the set $\{0, 1, 2, \ldots\}$ of non-negative integers. Number theorists usually mean the set $\{1, 2, 3, \ldots\}$ of positive integers. In this course, NT will usually refer to positive integers rather than natural numbers to avoid confusion.

Course Material

Some of the ML material for this course carries the course code *M335*, rather than *M381*. This is because this half originally formed part of the larger course *M335*. We hope that this will not cause any confusion. The Number theory half was rewritten for 1996.

(i) **Components of the Number Theory half of the course**

(a) Eight units, as follows.

Unit 1 Foundations
Unit 2 Prime Numbers
Unit 3 Congruence
Unit 4 Fermat's and Wilson's Theorems
Unit 5 Multiplicative Functions
Unit 6 Quadratic Reciprocity
Unit 7 Continued Fractions
Unit 8 Diophantine Equations

(b) Eight *Bookmarks*, one for each unit, which provide the unit notes.

(c) *Handbook* (which can be taken into the examination).

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(ii) **Components of the Mathematical Logic half of the course**

(a) Eight units, as follows.

Unit 1 Turing Machines and Computability
Unit 2 Abacus Machines
Unit 3 Recursive Functions
Unit 4 Church's Thesis
Unit 5 Formal Systems
Unit 6 Quantifier Logic
Unit 7 Formal Number Theory
Unit 8 Gödel's Incompleteness Theorems

(b) *Turing Machine Kit*

(c) *Exercise Booklet*

(d) *Handbook* (which can be taken with you to the examination).

(e) *Mathematical Logic Guide* (containing Unit Notes). This is Section 2 of this booklet.

(f) Three Audio Cassettes

In addition, there will be the following items: *Study Calendar, Assignment Booklets, Specimen Examination Paper, Solutions to the Specimen Examination Paper* and *Stop Presses.* The *Stop Presses* may contain errata and amendments to the course material, and should be read as soon as you receive them.

Assessment

The assessment of the course is by TMA and end-of-year examination. The TMAs are as follows.

	Material covered	
	NT	ML
TMA01	*Units 1* and *2*	*Unit 1*
TMA02	*Units 3* and *4*	*Units 2* and *3*
TMA03	*Units 5* and *6*	*Units 4* and *5*
TMA04	*Units 7* and *8*	*Units 6* and *7*

Unit 8 of ML is not assessed by TMA but is assessed in the examination.

Your overall continuous assessment grade is based on your scores in these four TMAs with substitution allowed for one TMA. (See the current *Student Handbook* for details of substitution.) Further information on the examination, including a specimen examination paper, will be sent to you later in the year. You may take the two *Handbooks* into the examination, but no other course material, so obviously it would be a good idea to get used to using these early in your study.

(Please note, however, that if you wish to take these *Handbooks* into the examination room, you may only ring, highlight and underline material, and correct errors.)

Calculators

You may well find a calculator useful during your study of NT. However you will not be allowed to use a calculator in the examination. The examination paper will be designed so that all calculations can be done using pen and paper.

Study Pattern

We have suggested, in the *Study Calendar*, that you study the units in the order NT1, then ML1, then NT2, then ML2, etc. This is mainly because we think that, on balance, you will benefit from studying both topics concurrently. But obviously, as one major constraint is what is contained in each of the TMAs, you may choose a variant of this order of study which matches the TMAs. However, the other constraint is what your tutor assumes you should have studied before tutorials, and here the order which we have suggested is likely to be the one assumed by your tutor.

Guides to Studying the Units

Each of the Number Theory course units is accompanied by a *Bookmark* which contains the study guide for the unit. Further general advice on studying the Number Theory half of the course is given in the introduction of *Unit 1* of Number Theory. The study guide for the Mathematical Logic half of the course appears as the next section of this *Course Guide.*

Both halves of the course have additional exercises for further practice; these appear as a section at the end of each unit in Number Theory but are in a separate *Exercise Booklet* for Mathematical Logic.

Tutorials

Regional arrangements will be made to provide a limited amount of face-to-face and/or telephone tuition for the course. If you are studying the units in a non-standard order, make sure to check with your tutor what material he or she thinks is likely to form the basis of tutorials.

Audio Cassettes (Mathematical Logic only)

There are a number of audio-cassette tapes that form an integral part of the ML half of the course course. At various points in the units, you will be instructed to listen to a cassette tape. As often as not, these tapes will themselves contain instructions on what work you should do next, as well as commentaries and advice on the mathematics.

		Associated Sections of ML
Tape 1	side 1	1.0 of *Unit 1*
	side 2	1.3 of *Unit 1*
Tape 2	side 1	2.3 of *Unit 2*
	side 2	6.3 of *Unit 6*
Tape 3	side 1	6.5 of *Unit 6*
	side 2	8.6 of *Unit 8*

2 MATHEMATICAL LOGIC GUIDE

The purpose of this half of the course is to teach you all you need to know to understand Gödel's (First) Incompleteness Theorem. This theorem is of great philosophical importance, as it tells us that there are surprising theoretical, as well as practical, limitations on what we can prove in mathematics.

The theorem requires us, *inter alia*, to show how mathematical proof is essentially a mechanical process, which in turn requires us to look at what can be achieved by mechanical processes. The first four units of this half look at three notions of what it means for the values of a function to be computable, and show that they are equivalent. The next three units look at a system of formal proof for the arithmetic of the natural numbers, while the last unit joins everything together in its discussion of Gödel's Theorem.

In its discussions of computability, recursive functions, formal proof, axiom systems and Gödel's Incompleteness Theorem, this half also provides a starting point for study of a variety of the topics that make up modern Mathematical Logic.

The Set Textbook

This half of the course makes use of the set textbook by Boolos and Jeffrey, *Computability and Logic* (Cambridge University Press, 1980, 2nd edition). You are likely to be using the 3rd edition of this book, which is fine. (If you are using an older edition, note that the 1st edition of this book will not suffice, especially for our *Unit 4*. Note also that the original printing of the second edition has an error, to which we refer in our list of Errata.) We ask you to read passages and attempt problems from this book as part of your study of *Units 1–4* and *8*. Although our course covers less than half of the material in the book, we feel that after studying it you will be in a position to read and appreciate much more of the book.

Problems in the Units

It is intended that you will attempt all the problems in the units. If you are short of time, you are occasionally advised that you need only read through a problem, and look straightaway at our solution in case it contains some teaching point.

Some further exercises on *Units 3–7* appear in the *Exercise Booklet*. These have been found especially helpful for *Units 3* and *4*.

The Handbook

The *Handbook* is intended to be of use in the examination. Some examination questions specifically allow you to quote results from the *Handbook*, so familiarize yourself with its contents (which refer to *Units 3–8*). Remember that should you wish to take the *Handbook* into the examination room, you may only ring, highlight and underline material, and correct errors.

Objectives of the Units

The objectives, especially those that are assessable, of each unit are listed as the summary of the unit. This appears at the end of the unit.

The Turing Machine Kit

This kit is intended for use with *Unit 1*. Instructions on its use are given on a cassette, Tape 1, side 2, which accompanies Section 1.3 of *Unit 1*. When you use the kit, you will find it helpful to clear a space on a desk or table top of dimensions at least 3′ wide by 1′6″ deep, on which you can lay out the kit.

Notes on the Units

Below, you will find some comments on the relative importance of the various sections of the units. If you are short of time, concentrate on the starred sections and on the problems that we list as most important, and just read quickly through all the less important problems (and their solutions) and the other sections of the units.

Unit 1

1.0 This section centres around a cassette (Tape 1, side 1) which gives an historical introduction to, and motivation for, the course. It is non-assessed but, we hope, very interesting. A short section.

*1.1 A short section. Most important problems: 3 and 4.

1.2 A short reading section, motivating the first four units.

*1.3 The main and most important section of the unit — it may well take you five or so hours to study. It includes a cassette (Tape 1, side 2) explaining how Turing machines work and are talked about, and how to use the Turing machine kit; this tape may, by itself, take in the order of $1\frac{1}{2}$ hours to study. When we first look at Turing machines, we encourage you to use your kit to perform calculations. Later on, when doing the problems in this section, we feel you should get used to doing these calculations on paper and recording what is called the 'sequence of configurations' of a computation. This skill will be needed later in the course. Most important problems: 1 to 8.

1.4 The objectives of the unit. Refer to this when you want to measure your progress.

1.5 This is an appendix giving you further practice in the sort of problems that might come up in the TMA, if you feel you need it.

Unit 2

2.0 Very short.

2.1 An interesting look at the Halting Problem, but if you are already short of time, just read quickly through it, taking note of the point in Note (v) on page 31, on 'Plugging together'.

*2.2 The main section of the unit, teaching you to run and program abacus machines, and requiring a lot of work, say five hours. Most important problems: 4 to 11 and 16.

*2.3 The cassette (Tape 2, side 1), which gives an illustration of how to show that a simple abacus computable function is Turing computable, is very important. The reading passage that follows, proving the general result that abacus computable functions are Turing computable, is optional. If you are short of time, drop the optional passage and listen to the tape without taking time to devise the Turing machines it requires.

2.4 Objectives.

Unit 3

*3.0 A short but important reading passage.

*3.1 The main section of the unit. The only way to get to grips with the definition of primitive recursive function is to work seriously through the problems in this section, so you may spend a long time, say five hours, on them. Most important problems: 2, 3, 6 and 9 to 16.

*3.2 A straightforward reading section, showing that every primitive recursive function is abacus computable. Most important problems: 1 and 2.

3.3 A reasonably short section, discussing minimization, the extra rule that can be used to construct recursive rather than primitive recursive functions. From the point of view of the course, minimization is of more theoretical than practical importance, so do not get bogged down in this section. Most important problems: 1 and 2.

3.4 Objectives.

Unit 4

4.0 Very short.

*4.1 An important section (especially from the point of view of assessment), showing how wide the class of recursive functions is, as well as proving that certain special functions needed later are recursive. As with Section 3.1, you should have a good try at the problems, so this section may take you three hours to study. Most important problems: 1 to 5, 7, 9 and 10.

*4.2 A reasonable evening's work, giving some of the (easier) details of how one shows that Turing computable functions are recursive. Most important problems: 6 to 8, 10 and 11.

4.3 The nastier details of the proof that Turing computable functions are recursive. This section is non-assessed, so do not get bogged down trying to understand the detail, especially if you are short of time.

4.4 A short reading section.

4.5 Objectives.

Unit 5

5.0 Introduction.

*5.1 A long-looking section, but, we hope, reasonably easy to study, describing the formal language for number theory, viz. the basic symbols and the rules for constructing the meaningful statements of the language. Problems 1 to 4 are important but straightforward.

*5.2 A short reading section, motivating the idea of a formal proof and describing the first rule of proof, the Rule of Assumptions.

*5.3 This important section describes how to interpret the logical connectives using truth tables, and hence how to assign a truth value to many formulas. The important ideas of logical consequence and tautological consequence are also introduced. Most important problems: 1 to 4.

*5.4 This important section describes the two rules of proof, the Tautology Rule and Conditional Proof, and you get down to actual formal proofs. Most important problems: 1 to 3. Note that there are many more problems on formal proofs to come in *Units 6* and *7*, where further rules of proof are available.

5.5 Objectives.

5.6 An optional and non-examinable section, on how to construct a Turing machine to test whether a string is a term. Drop this section if you are short of time.

Unit 6

*6.0 A very short introductory section, but note the remarks on the interpretation of $\forall$ and $\exists$.

*6.1 A short technical section, on free and bound variables, of considerable importance in the rest of the unit. However, the significance of these ideas may not hit you until you have seen them used in later sections. Most important problem: 1.

*6.2 An important but short section, in which the Universal Quantifier Elimination Rule is described. This rule, and others later, use the important idea of a term which may be freely substituted for a variable in a formula. Most important problems: 1 to 3.

*6.3 A long section, accompanied by side 2 of Tape 2, looking at proofs involving the rules introduced in *Unit 5* and those involving the universal quantifier. It is important to take the time to understand why the formal proofs are correct, as well as how to set about constructing them. Most important problems: 4 and 5.

*6.4 A short section, describing the Existential Quantifier Elimination Rule. Most important problem: 1.

*6.5 A long section, accompanied by side 1 of Tape 3, on proofs involving all the quantifier rules. The method of proof by contradiction in Example 4 is very useful. Most important problems: 3 and 6.

*6.6 A short section describing the two rules for handling $=$. Do not worry overmuch about the technicalities in the statement of the Substitution Rule. In practice, as in Examples 2 and 3, its use is easy, but note the errata (for *Unit 7*) on page 10 of this Guide. Most important problem: 1.

6.7 A very short reading section. The Correctness Theorem will be exploited in *Unit 7*.

6.8 Objectives.

6.9 A brief reminder on the hints for finding formal proofs given on the cassette tapes.

Unit 7

7.0 Very short.

*7.1 A short but important section, describing how we can interpret the symbols of our language, the axioms of the system Q, and checking that these axioms are true in the standard interpretation. Most important problem: 1.

*7.2 A long section (if you are being thorough), on formal proofs from the axioms of Q. Most important problems: 1, 4(ii), 4(iii)and 5.

*7.3 An important section, using the Correctness Theorem to show when sentences are *not* theorems of Q. Most important problems: 1 and 2.

7.4 This section describes the representability of a function in Q, an important ingredient in the proofs in *Unit 8*. Just read through the solutions to the problems if short of time, although Problems 2, 3 and 4 give useful extra practice in writing formal proofs.

7.5 Objectives.

Unit 8

This unit is not assessed by TMA, but is examinable.

8.0 Very short.

*8.1 A short section, describing gödel numbering.

*8.2 A longer reading section, describing Gödel's Diagonal Lemma. Skip the proofs of Lemmas A and B, the solution to Problem 3, and even the proof of the Diagonal Lemma itself if you are short of time (or energy), and go on to Section 8.3 to see how this latter Lemma is used.

*8.3 Leibniz's Question is answered in this fairly short reading section. Make sure to remember the meanings of 'consistent', 'satisfiable' and 'definable set'.

*8.4 Another short reading section, leading to a proof of Gödel's First Incompleteness Theorem. Make sure you understand the meanings of 'axiomatizable', 'recursively enumerable' and 'complete'.

8.5 A short section, introducing the axioms for Peano Arithmetic.

8.6 This section looks at Gödel's Second Incompleteness Theorem, and is accompanied by side 2 of Tape 3, which comments on all the examinable results in the unit.

8.7 A list of books for further reading.

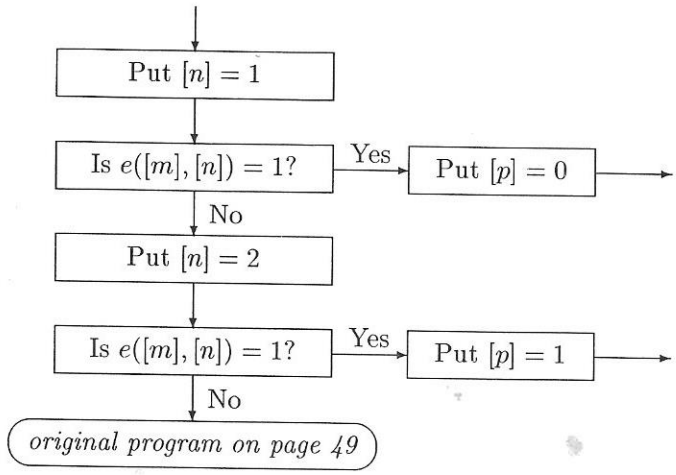
8.8 Objectives.

3 ERRATA AND ADDENDA: MATHEMATICAL LOGIC

(i) Errata in Units 1–4

Page	*Location*	*Correction*
21	Solution 3(ii)(c)	At the end of the sequence of configurations, replace Input 6 / Output 1 with Input 6 / Output 0
35	Register table	The $[m]$ column should read $\not{3}$, $\not{2}$, $\not{1}$, 0 rather than $\not{3}$, $\not{2}$, $\not{2}$, 0
48	Solution 16(ii)	Add the box $\boxed{0 \longrightarrow m}$ $\downarrow$ at the bottom of the program.

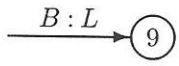
Page	*Location*	*Correction*
49–50	Solution 16(v)	The procedure and program given are incorrect. The original outline solution on page 49 incorrectly gives 1 as a prime and 2 as a non-prime! (It correctly gives 0 as a non-prime and identifies precisely all primes greater than 2.) To correct the outline program on page 49, start the program with tests to check whether $[m] = 1$ or 2, e.g. using the function e of Problem 16(iii) on page 46 of the unit, as follows.



Of course, when amending the full program on page 50, a copy of $[m]$ needs to be taken before using the block diagram for the function e, to prevent its value being lost.

54 Frame 16

There should be an arrow coming out of state 8, moving the scanning head one place to the left, to stop in a state 9, e.g.



62 Line −9

The equation should be

$$\text{prod} = \text{Pr}[z, \text{Cn}[\text{sum}, \text{id}_1^3, \text{id}_3^3]].$$

78 Problem 5, line 2

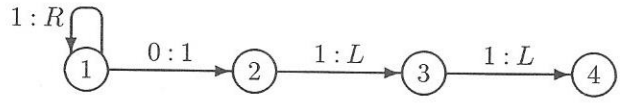
Before 'conditions' insert 'primitive recursive'.

86 Note (iv)

This should refer to **BJ** page 91, line −1, not line 1.

86–7

The tables for $a(x, y)$ and $q(x, y)$ at the bottom of page 86, and the remarks about these tables in the first paragraph on page 87, refer to the machine



given in Example 8.3 of **BJ**, 2nd edition 1980. As this machine does not, as was claimed in this edition, compute the successor function, the machine has been revised in **BJ**, 2nd edition 1980, reprinted 1982, and in the 3rd edition 1989, which is the version most students will have. The tables of values $a(x, y)$ and $q(x, y)$ would be given as below.

$a(x,y)$	y = 0	1
1	1	3
x 2	3	2
3	0	1

$a(x, y) = y$ otherwise

$q(x,y)$	y = 0	1
1	2	1
x 2	3	2
3	0	0

$q(x, y) = 0$ otherwise

(ii) Errata in Units 5–8

Page	*Location*	*Correction*
48	Example 1, line (4)	'UE, (2)' should be 'UE, (3)'
58	line 2	'UE, (2)' should be 'UE, (1)'.
73	Solution 4(ii)	Although the proof is itself correct, the assumption on line (13), $\forall x - x' = \mathbf{0}$, is not in fact an axiom of Q, so that one cannot immediately deduce $\vdash_Q -(\mathbf{0}' + \mathbf{0}') = \mathbf{0}'$ from line (17) of the proof. However, it is easy to show that $\vdash_Q \forall x - x' = \mathbf{0}$ and incorporate this in the proof, or we might replace lines (13) onwards by the following.

13	(13)	$\forall x - \mathbf{0} = x'$	Ass
13	(14)	$-\mathbf{0} = \mathbf{0}'$	UE, (13)
1, 2, 3, 7, 13	(15)	$-\mathbf{0} = \mathbf{0}$	SR, (12), (14)
	(16)	$\mathbf{0} = \mathbf{0}$	II
1, 2, 3, 7, 13	(17)	$(\mathbf{0} = \mathbf{0}\ \&\ -\mathbf{0} = \mathbf{0})$	Taut, (15), (16)
1, 2, 7, 13	(18)	$((\mathbf{0}' + \mathbf{0}') = \mathbf{0}' \longrightarrow (\mathbf{0} = \mathbf{0}\ \&\ -\mathbf{0} = \mathbf{0}))$	CP, (17)
1, 2, 7, 13	(19)	$-(\mathbf{0}' + \mathbf{0}') = \mathbf{0}'$	Taut, (18)

As the assumptions 1, 2, 7 and 13 are now genuine axioms of Q, $\vdash_Q -(\mathbf{0}' + \mathbf{0}') = \mathbf{0}'$.

73	Solution 4(iii), line (1)	'$\forall x(x + \mathbf{0}) = \mathbf{0}$' should be '$\forall x(x + \mathbf{0}) = x$'
90	Solution 3, proof at bottom of page	The use of SR on line (5) is incorrect. The proof needs two extra lines between lines (3) and (5). We shall call these lines $(3\frac{1}{2})$ and $(4\frac{1}{2})$ to save relabelling the rest of the proof! Replace lines (4) and (5) with the following.

	$(3\frac{1}{2})$	$x_2 = x_2$	II
4	(4)	$x_2 = \mathbf{p}_2$	Ass
4	$(4\frac{1}{2})$	$\mathbf{p}_2 = x_2$	SR, $(3\frac{1}{2})$, (4)
1, 4	(5)	$A(\mathbf{p}_1, x_2)$	SR, (1), $(4\frac{1}{2})$

(iii) ML Handbook

Page	*Location*	*Correction*
4	truth table (e)	'$\phi \longleftrightarrow \psi$' should really be enclosed in parentheses, i.e. '$(\phi \longleftrightarrow \psi)$'.

(iv) Erratum in ML Exercise Booklet

Page	*Location*	*Correction*
7	Solution 3.4(iii):	The definition of $h(x)$ should read $$h(x) = \sum_{n<x} l(n).$$

(v) Extra Notes for ML

Unit 1 page 7

It helps to define here what is meant by 'function of r arguments'. If f is a function that takes as argument a pair (x_1, x_2) of numbers (so that the domain of f consists of a set of pairs of numbers), we say that f is a function of 2 arguments, and call x_1 the *first argument* of the function and x_2 the *second argument* of the function.

Similarly, if f is a function that takes as argument an ordered r-tuple $(x_1, x_2, \ldots, x_r)$ of numbers, we say that f is a function of r arguments, and call x_i the ith argument of f, for each $i = 1, 2, \ldots, r$. (Note when r is small, e.g. 1, 2 or 3, it is common to use x, y, z for the arguments instead of x_1, x_2, x_3.)

Example The function $f = N \times N \times N \longrightarrow N$ defined by

$$f(x, y, z) = (x \cdot y) + z$$

is a function of three arguments: x is the first argument, y the second and z the third.

Unit 4 page 77 and BJ page 85

BJ's discussion on page 85 of Definition by Cases refers to functions of two arguments, but applies perfectly well to functions of one or three or more arguments.

For instance, if $C_1, \ldots, C_n$ are mutually exclusive, collectively exhaustive conditions on natural numbers x, with characteristic functions $c_1, \ldots, c_n$ (which thus each have one argument x), then the function f of one argument defined by

$$f(x) = \begin{cases} g_1(x), & \text{if } C_1, \\ \vdots & \\ g_n(x), & \text{if } C_n, \end{cases}$$

where each condition C_i is primitive recursive, and each function g_i is primitive recursive, will also be primitive recursive. The justification of this is similar to the argument on lines 16–20 of **BJ** page 85, establishing that the formula

$$f(x) = g_1(x)c_1(x) + \ldots + g_n(x)c_n(x)$$

holds for all x. As f is then obtained by adding and multiplying primitive recursive functions, f is itself primitive recursive.

Unit 6, Pages 60–1

There is a maddening feature of the Substitution Rule, not specifically referred to within the unit, which is that the order of the terms τ_1 and τ_2 in the statement of the rule (on page 60) is critical for a correct use of the rule. For instance, the first three lines of the proof in Example 2 on page 60, reprinted below, essentially prove that $x = y \vdash x' = y'$.

1	(1) $x = y$	Ass
	(2) $x' = x'$	II
1	(3) $x' = y'$	SR,(1),(2)

But to prove that $y = x \vdash x' = y'$, which is indeed a theorem, it is not enough to replace the $x = y$ in line (1) above by $y = x$. This would give the following 'proof':

1	(1) $y = x$	Ass
	(2) $x' = x'$	II
1	(3) $x' = y'$	SR,(1),(2)

The use here of SR on line (3) is incorrect. The order of the terms on line (1) means that, using the symbolism of the statement of SR, τ_1 is y, the term to the left of the $=$ in $x = y$, not x; and there are no occurrences of this τ_1 in the formula on line (2) for which τ_2, which would be x, can be substituted. (Remember that the formalism of the rules of proof is designed so that a Turing machine could check the correctness of a proof. So the rules are chosen to be a fairly minimal subset of all possible valid rules, for reasons of economy. The down side is the sort of restriction on use which bans the last 'proof'.)

As an exercise you might like to dream up a replacement for line (2) of the previous 'proof' to turn it into a correct three line proof of $y = x \vdash x' = y'$.

Note that this feature of SR is what underlies the error in Solution 3 on page 90 of *Unit 7*, which is corrected earlier in this Course Guide.

Unit 7 pages 66–7

The definition of a non-standard interpretation of the arithmetical symbols is not fully spelt out here, although it is later, in Section 7.3.

An interpretation of the symbols must certainly specify a domain U, but must also explain how to interpret the symbols $'$, $+$, $\cdot$, $\mathbf{0}$. Remembering that terms, which are built up from these symbols (and the variables), simply stand for objects in the domain U, all we require of the interpretation of $'$, $+$, $\cdot$, $\mathbf{0}$ is that

$'$ is interpreted by a function $U \longrightarrow U$,
$+$ is interpreted by a function $U \times U \longrightarrow U$,
$\cdot$ is interpreted by a function $U \times U \longrightarrow U$,
$\mathbf{0}$ is interpreted by a specified element of U.

In Section 7.1 all the non-standard interpretations happen to interpret $'$, $+$, $\cdot$, $\mathbf{0}$ in the standard way and are non-standard purely because their domains are sets different from N. In Section 7.3 you will see interpretations in which, for example, $+$ is interpreted in a non-standard way.

A further example of a non-standard interpretation is the following.

Let $U = N$, the set of natural numbers.
Interpret
$+$ in the usual way,
$'$ by the function $f : N \longrightarrow N$, where $f(n) = n^2$,
$\cdot$ by the function $g : N \times N \longrightarrow N$, where $g(n, m) = 13$ for all n, m,
$\mathbf{0}$ by the specified number 42.

So, for example, the interpretation of the term $(\mathbf{0} \cdot \mathbf{0})'$ is the natural number $f(g(42, 42))$, i.e. 169.

Unit 8 pages 102–3

Two amusing and challenging books of logic puzzles (in paperback) which lead to discussions of Gödel's Incompleteness Theorem and Leibniz's Question, both by Raymond Smullyan, are *What is the Name of this Book?* (Pelican, 1981), and *The Lady or the Tiger* (Pelican, 1983).

A very interesting (but rather expensive) book has been published about Alan Turing with good detail of his work on Turing machines, as well as on the Enigma codes and the development of computers. It is *Alan Turing: the Enigma*, by Andrew Hodges, published by Burnet Books (1983).

An advanced, but nonetheless very readable, book illustrating further some of the strong connections between the number theory and mathematical logic encountered in this course is *Logical Number Theory*, Volume 1, by Craig Smorynski, Springer–Verlag (1991).

Another advanced but still very readable book on computability and logic is *Computability: Computable Functions, Logic and the Foundations of Mathematics*, by R.L. Epstein and W.A. Carnielli, Wadsworth (1989).

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